DIFFUSION MODEL OF DUST LIFTING BEHIND A SHOCK WAVE

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Abstract

Investigations of shock induced boundary layers over dust deposits are of practical importance, in particular with regard to fire and explosion safety problems. In this paper a diffusion model of dust lifting behind a shock wave is proposed. The process of dust lifting is considered as isothermal turbulent mixing of two liquids of different densities and velocities, i.e. shocked gas and a dust layer. The diffusion model shows a linear increase of the dusty boundary layer height depending on the distance to the shock front, which is in agreement with experimental data. Calculated particle density and velocity profiles are also in a satisfactory agreement with experimental observations. The model allows to estimate the mass entrainment rate of dust particles into an air flow behind a shock wave.

Introduction

Investigation of shock induced boundary layers over dust deposits are of practical importance, in particular with regard to fire and explosion safety problems (Refs.1–6). A shock wave generated as a result of an accidental explosion may disperse a sedimentary dust layer. If there exists additionally any ignition source or the shock wave intensity is high enough to ignite dust particles (coal particles in mine galleries, grain dust in elevators, etc.) the explosion process may be enhanced dramatically.

The process of dust lifting in the flow behind a shock wave was studied experimentally (Refs.2–4, 6) and theoretically (Refs.1, 5). The new impact to the problem is connected with the development of modern diagnostic techniques (Ref.8) which allow to measure dust density and velocity distributions over the layer. It follows from experimental observations that the height of the dusty boundary layer increases linearly with the distance from the shock front. This circumstance indicates an analogy between the dust lifting process behind a shock wave.
and the process of turbulent mixing in a shear layer (Ref. 7). As is known, in the latter case there exists the self-similar velocity distribution and the width of the mixing layer increases linearly with the distance. In this paper a diffusion model of dust lifting behind a shock wave is proposed and a comparison with experimental observations is made.

**Formulation**

Let us consider the process of dust lifting behind a shock wave as isothermal turbulent mixing of two liquids of different densities and velocities i.e. the shocked gas and the dust layer. It means that we consider dust particles as a certain passive additive exerting no influence on the turbulence characteristics in the mixing layer. This assumption is apparently valid for relatively small particles with small characteristic times of velocity relaxation. Let the shock wave propagates at the constant velocity $u_1$. In the shock fixed frame of coordinates the dust layer has initially the velocity $u_1$ while the air flow has the velocity $u_2 = u_1 - u_\infty$, where $u_\infty$ is the gas particle velocity (Fig.1).

The mixing process we describe by the following set of differential equations

\[
\text{Conservation of mass} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]

\[
\text{Conservation of momentum} \quad \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial x} [ \rho \frac{\partial u}{\partial x} ] + \frac{\partial}{\partial y} [ \rho \frac{\partial u}{\partial y} ] - (1)
\]

\[
\text{Turbulent diffusion} \quad u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} = \frac{\partial}{\partial x} [ \rho \frac{\partial \rho}{\partial x} ] + \frac{\partial}{\partial y} [ \rho \frac{\partial \rho}{\partial y} ]
\]

Here $l = \frac{\tau_{\text{mix}}}{\tau_{\infty}}$ is the Tollmien's mixing length (Ref.7), $c$ is the constant, $\rho$ is the mixture density, $u$ and $v$ are respectively longitudinal and transverse velocity components. Boundary conditions for Eqs.(1) we formulate on the basis of physical considerations: at $y = \infty$ $\frac{\partial u}{\partial y} = 0$, $\rho u = \rho_1 u_1$, $\rho = \rho_1$ (we consider an infinitely deep dust layer); at $y = -\infty$
\[ \frac{\partial u}{\partial y} = 0, \quad \rho u = \rho_2 u_2, \quad \rho = \rho_2. \] For the transverse velocity component in dust deposit we assume that at \( y = \infty \) \( \rho v = 0 \).

The fact that the values of \( \rho, u, \) and \( v \) remain constant at the boundaries allows us to assume that there exists the aerodynamic similarity of density and velocity distributions at various flow cross sections. Let us introduce the dimensionless similarity coordinate \( \phi = y/ax \) (here \( a = (2c^2)^{1/3} \) is the constant) and the dimensionless stream function \( F(\phi) \) such that
\[ \rho u = \rho_1 u_1 F', \quad \rho v = \rho_1 u_1 a(\varphi F' - F) \]
The dimensionless mixture density is
\[ \rho = \rho_1 k(\phi) \]
After substitution of these transformations into Eqs.(1) the equation of continuity is satisfied automatically while the other equations take the form
\[ -F = F'''' - \left( \frac{k}{k'F'/k} \right)' \quad (2a) \]
\[ -F = \left[ k(F'/k) \right]' \quad (2b) \]
where prime denotes differentiating on \( \phi \). The aim of further transformations of Eqs.(2) is to separate variables \( k \) and \( F \).

Equating r.h.s. of Eqs.(2) and denoting \( \mu(\phi) = k(F'/k)' \) one obtains
\[ k''/k' = \mu'/\mu \]
with the solution \( \mu = C_1 k' \), or in the initial variables:
\[ k'(C_1 k + F') - F''k = 0 \quad (3) \]
where \( C_1 \) is the integration constant. The further transformation \( \rho(\phi) = C_1 k \) gives to Eq.(3) the form of II order Abel equation in respect to the variable \( p \):
\[ p' (p + F') - F''p = 0 \quad (4) \]
Again, the substitution \( z = (p + F')^{-1} \) transforms Eq.(4) into the I order Abel equation relatively to the variable \( z \):
\[ z' = -2F''z^2 + F'F'z^3 \]
At last, the substitution \( q = -0.5 F'z \) results in the equation with separated variables
\[ q' = (4F''/F')(q^3 + q^2 + (1/4)q) \quad (5) \]
The solution of Eq.(5) in initial variables is given by
\[ \ln (-C_1 k) - (F'/C_1 k) = C_2 \quad (6) \]
where \( C_2 \) is the second integration constant. It follows from Eqs.(2a) and (6) that
Differentiation of Eq. (7) one time gives

\[ F' = C_1^2 \kappa'' \]

On the other hand one obtains from Eq. (6)

\[ -F' = C_1 C_2 \kappa - \kappa C_1 \ln(-C_1 \kappa) \]

Combination of the two last equations gives the resultant equation for density distribution in the mixing layer

\[ \kappa'' = \kappa C_2 \ln(C_1 \kappa) \]

In order to determine the integration constants \( C_1 \) and \( C_2 \) we use Eq. (6) and the conditions at the boundaries of mixing layer, i.e. \( \varphi = \varphi_1 \) (the boundary in the dust layer): \( \kappa = 1, F' = 1; \varphi = \varphi_2 \) (the boundary in air): \( \kappa = \kappa_2, F' = \kappa_2 u_2/u_1 \). Then

\[ C_1 = \frac{1}{\ln \kappa_2}, \quad C_2 = \ln \frac{1 - u_2/u_1}{\ln \kappa_2} + \frac{1}{1 - u_2/u_1} \]

With due regard for Eqs. (9) the resultant Eq. (8) takes the form

\[ \kappa'' + \kappa \ln \kappa + \kappa \ln \left(\frac{\kappa_2}{u_2/u_1 - 1}\right) = 0 \]

Since the boundaries of the mixing layer are unknown in advance one should impose five boundary conditions on Eq. (10):

\[ \varphi = \varphi_1: \kappa = 1, \kappa' = 0, \kappa'' = -\varphi_1 \frac{\ln \kappa_2}{u_2/u_1 - 1} \]

\[ \varphi = \varphi_2: \kappa = \kappa_2, \kappa' = 0 \]

The condition for \( \kappa''(\varphi) \) follows from Eq. (7). When writing the boundary conditions (11) we used the assumption that the boundaries of mixing layer in respect to velocity and density are the same. If the solution of the problem (10), (11) has been obtained one may easily determine the velocity components:

\[ u/u_1 = F'/\kappa = 1 + \frac{\ln \kappa_2}{u_2/u_1 - 1} \ln \kappa \]

\[ \frac{\varphi F' - F}{\kappa} = \frac{u_2/u_1 - 1}{\ln \kappa_2} \left[ \frac{\varphi \ln(\kappa/\kappa_2)}{\kappa} + \kappa'' + \frac{\varphi u_2/u_1}{\kappa} \right] \]

When deriving expressions for \( v \), Eqs. (6), (7) and (9) were
used. Thus, Eqs.(10) to (12) define the problem completely. The solution of the problem depends on two dimensionless parameters: \( \kappa_2 = \rho_2/\rho_1, \ u_2/u_1 \) i.e. on the dust layer density and on the shock wave Mach number.

At \( \kappa_2 = 1 \) the problem has a trivial solution \( \kappa = 1 \); Eqs.(2a) and (2b) are identical and have the form
\[
F'''+F = 0
\]
Boundary conditions for calculation of the velocity field in this case follow from Eqs.(6), (7) and (11)
\[
\begin{align*}
\phi &= \varphi_1: & F &= \varphi_1, & F' &= 1, & F'' &= 0 \\
\phi &= \varphi_2: & F' &= u_2/u_1, & F'' &= 0
\end{align*}
\]
In the particular case \( u_2 = 0 \) one comes to a standard problem on the isothermal mixing of a plane incompressible liquid jet (Ref.7).

Results

Our attempts to find an analytical solution of Eq.(10) have failed and we integrated it numerically by the use of the standard Runge-Kutta method. Equation (10) was transformed to the set of first order differential equations
\[
\begin{align*}
\kappa' &= \nu, \\
\eta' &= - \kappa \ln \kappa + \kappa \ln(\kappa_2)/(u_2/u_1 - 1)
\end{align*}
\]  
with boundary conditions
\[
\begin{align*}
\phi &= \varphi_1: & \kappa &= 1, & \nu &= 0, & \eta &= - \varphi_1 \ln (\kappa_2)/(u_2/u_1 - 1) \\
\phi &= \varphi_2: & \kappa &= \kappa_2, & \nu &= 0
\end{align*}
\]
and the physical restriction
\[
\kappa_2 < \kappa < 1
\]
One may show that the problem defined by Eqs.(13), (14) has the unique solution.

The procedure of numerical solution is as follows. First we choose a certain value of \( \varphi = \varphi_1 > 0 \) and calculate boundary conditions at \( \varphi = \varphi_1 \). In this way the starting conditions for the solution are ensured. We use the constant integration step \( \Delta \varphi < 0 \) and control an approximation error at each step. The integration procedure is carried out up to the moment when one of the conditions \( \kappa = \kappa_2 \) or \( \nu = 0 \) is satisfied in the frame of restriction given by Eq.(14c). The further procedure reduces to
changes of the starting conditions (the value of \( q_1 \)) such that Eqs. (14b) become satisfied simultaneously. In such a way the values of \( q_1 \) and \( q_2 \), and hence density and velocity distributions were determined as functions of the parameters \( \kappa_2 \) and \( u_2/u_1 \).

When comparing the results of calculations with the experimental observations we have run into certain difficulties. Experiments of Ref. 6 indicate that the mass particle concentration at the surface \( y = 0 \) (Fig. 1) does not exceed 5 - 15 kg/m³ when the dust density in the layer interior is of the order of \( 10^3 \) kg/m³ and the shock wave Mach number \( M = 1.3 - 1.8 \). The solution of the problem defined by Eqs. (13), (14) for \( \kappa_2 = 2 \cdot 10^{-3} \) (\( \rho_1 = 800 \) kg/m³) and \( u_2/u_1 = 0.592 \) (\( M = 1.4 \)) gives \( \kappa(0) = 0.65 \) which is equivalent to the particle mass concentration of 520 kg/m³ at the surface.

Probably, the reason of such a discrepancy of the theoretical and experimental results consists in the essential schematization of the air - dust mixing process. In the model formulated above we did not take into account the process of particle collisions in the layer. The relatively low particle concentration at the layer surface indicates that only insignificant amount of particles participates in the mixing process. Thus, we come to a conclusion that the dust layer may be considered formally as a medium composed of the unmoving relatively heavy lattice, the cloud of non-inertial particles and a gas filling cavities in the lattice. Particles diffuse from the layer interior and support the mixing process. On the other hand, gas diffuses into a layer. The role of the lattice consists only in reducing the amount of particles participating in the process at any given time.

The solution of the problem (13), (14) shows that in order to attain the observed value of the particle mass concentration in the mixing layer one should assume that the particle mass concentration in a cloud is not more than 10-20 kg/m³. The solid curve in Fig. 2 shows the solution for \( M = 1.67 \) (\( u_2/u_1 = 0.465 \)) and \( \kappa_2 = 0.25 \) (\( \rho_1 = 10 \) kg/m³). Plotted along the x-axis is the particle mass concentration \( \rho_d = \rho_1 (\kappa - \kappa_2)/(1 - \kappa) \).
\( x_0 \). Plotted along the \( y \)-axis is the self-similar coordinate \( \phi \).

The height of the mixing layer in the air \( \delta = a \phi_2 x_0 \), where \( x_0 = tu_0 \) is the distance to the shock front, \( t \) is the time. Experimental observations in Ref.6 indicate that \( \delta = -(0.015 \pm 0.0045)x_0 \) for \( M = 1.3-1.8 \) and the particle size 7-150 \( \mu \)m. Table 1 shows values of \( \phi_1 \) and \( \phi_2 \) as well as the values of particle mass concentration at the surface \( y = 0 \). It follows from Table 1 that for \( M = 1.3-1.8 \) \( -\phi_2 \approx 1.18 \pm 0.14 \). On the basis of experiments (Ref.6) we may now conclude that the constant \( a \) in the definition of the self-similar coordinate \( \phi \) is \( a \approx 0.013 \pm 0.005 \). To recalculate the experimental dependencies \( \rho_a(y) \) in Ref.6 we use the following transformation: \( \phi = y/ax_0 \).

Shown in Fig.2 by points are the measured values of particle mass concentration in the mixing layer at \( M = 1.67 \) at different times after shock wave arrival. The agreement between the experimental results and the results of calculations (solid line) is satisfactory even for relatively large particles (100 \( \mu \)m). The considerable difference between the results is apparent only in the vicinity of the surface \( y = 0 \). Note, that by the proper adjustment of the \( \rho_1 \) value one may obtain better agreement nearby \( y = 0 \).

Shown in Fig.3a,b is the comparison of the normalized calculated density profiles in the mixing layer (solid lines) with experimental results of Ref.6. As before \( \rho_1 = 10 \text{ kg/m}^3 \) is assumed. The dashed lines represent the 7/10's power law model suggested heuristically in Ref.6:

\[
\frac{u}{u_\infty} = \frac{1}{1 + (y/\delta)^{0.7}}
\]  

Figure 4 shows a comparison of the calculated normalized profile of the longitudinal velocity component (solid line) with the experimental results of Ref.6. The dashed line represents the power law (15).

Shown in Fig.5 is the summary plot of calculated (solid line) and measured (Ref.6) data on dust entrainment rate \( m \) vs. the shock wave overpressure \( \Delta p_\infty \).
Conclusion

The diffusion model of dust lifting behind a shock wave is proposed. The model shows a linear increase of the dusty boundary layer height depending on the distance to the shock front, which is in agreement with numerous experimental observations. The model allows to estimate the mass entrainment rate of dust particles into an air flow behind a shock wave.

References


Table 1. Boundaries of the mixing layer behind a shock wave propagating above a dust deposit with $\rho_1 = 10$ kg/m$^3$.

<table>
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<th>$M$</th>
<th>1.3</th>
<th>1.4</th>
<th>1.5</th>
<th>1.6</th>
<th>1.7</th>
<th>1.8</th>
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<tr>
<td>$\varphi_1$</td>
<td>0.528</td>
<td>0.567</td>
<td>0.597</td>
<td>0.622</td>
<td>0.642</td>
<td>0.660</td>
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<tr>
<td>$\varphi_2$</td>
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<td>-1.17</td>
<td>-1.23</td>
<td>-1.27</td>
<td>-1.32</td>
</tr>
<tr>
<td>$\rho_d(0)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>kg/m$^3$</td>
<td>6.70</td>
<td>6.74</td>
<td>6.76</td>
<td>6.78</td>
<td>6.79</td>
<td>6.79</td>
</tr>
</tbody>
</table>
Fig. 1 The sketch of the dust lifting model behind a shock wave.

Fig. 2 The comparison of the calculated (solid line) and measured (points of Ref. 6) particle density distribution in the mixing layer behind a shock wave of $M = 1.67; \rho_1 = 10 \text{ kg/m}^3$. Particle diameter: 
- $7 \mu m$ (t = 4 ms, 10 ms)
- $100 \mu m$

Fig. 3 The comparison of the calculated (solid lines) and measured normalized particle density distribution in the mixing layer at $M = 1.3-1.8$ and different times of shock wave arrival (from 2 to 15 ms, Ref. 6). Dashed line represents 7/10's power law suggested in Ref. 6.

a) particle diameter 7 \mu m, b) 100 \mu m
Fig. 4 The comparison of the calculated (solid line) and measured (Ref. 6) normalized particle velocity distribution in the mixing layer at $M = 1.38$ and particle size 7 $\mu$m at times from 2 to 8 ms after shock arrival. Dashed line represents 7/10's power law suggested in Ref. 6.

Fig. 5 The comparison of the calculated (solid line) and measured results for dust entrainment rate depending on the shock overpressure; $\rho_1 = 10$ kg/m$^3$. 